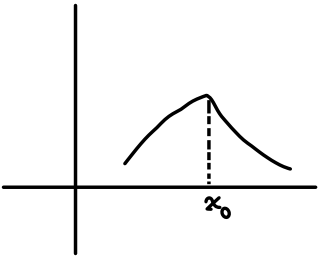
A function continuous on a closed interval enjoys many properties, and the graph of such a function consists of a single piece that can be traced without lifting the pencil off the paper.

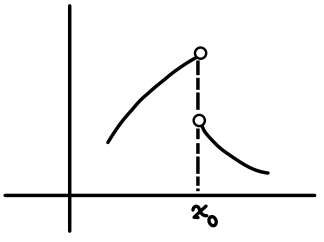
A function f, defined on a closed interval $[a, b]$, is said to be <u>continuous</u> at an interior point $x_o \in (a, b)$ if

$$\lim_{x \to x_o} f(x) = f(x_o) \qquad (1)$$

Continuity of f at the end points a and b is defined by (1) by the use of right and left limits at a and b respectively. A function which is not continuous at a point $x_o \in [a, b]$ is said to be <u>discontinuous</u> at x_o.



Graph of a function continuous at x_o



Graph of a function discontinuous at x_o

A function $f: \mathbb{R} \to \mathbb{R}$, given by a rule, is discontinuous at a point x_o at least in one of the following cases:

1) f is not defined at x_o (<u>Missing point discontinuity</u>)

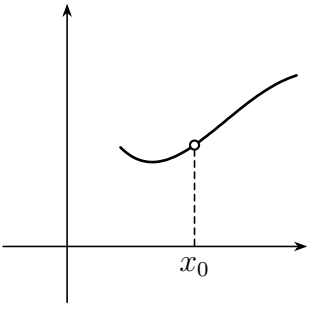
2) f is defined at x_o but not in a deleted neighborhood[1] $N(x_o)$ of x_o. (<u>Isolated point discontinuity</u>)

1) Some authors regard f continuous at an isolated point, in a deleted neighborhood of which f is undefined.

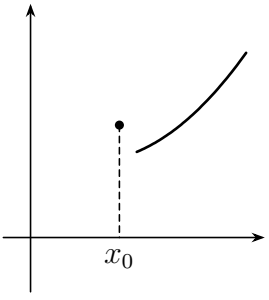
3) $\lim f(x)$ does not exist at x_0 (jump discontinuity)

4) $\lim f(x)$ exists but this limit is not equal to $f(x_0)$
(Removable discontinuity)

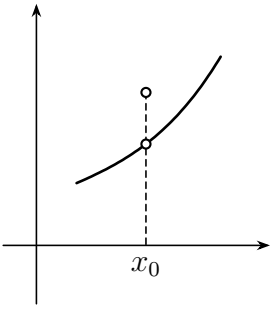
Then we have the following types of discontinuities as illustrated by graphs:



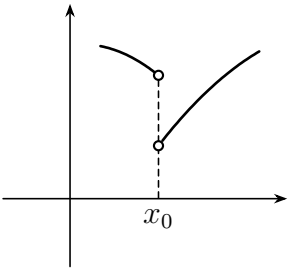
Missing point discontinuity
or removable discontinuity



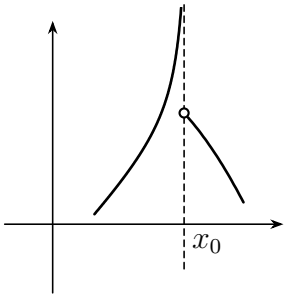
Isolated point
discontinuity



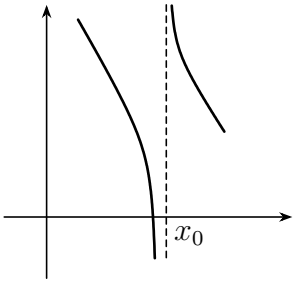
Finite jump
discontinuity



Finite jump
discontinuity



Infinite jump
discontinuity



Infinite jump
discontinuity

Example 1. Why the following functions are not continuous at indicated points, and what is the type of discontinuity?

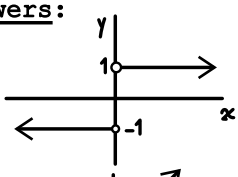
1. $f(x) = \dfrac{|x|}{x}, \quad x_0 = 0$

2) $g(x) = \begin{cases} 1 & \text{when } x = -1 \\ x & \text{when } x \geq 0 \end{cases}$, $x_o = -1$

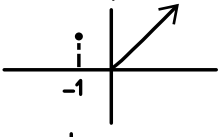
3) $h(x) = \dfrac{x - 2}{x - 2}$, $x_o = 2$

4) $k(x) = \dfrac{1}{x - 1}$, $x_o = 1$

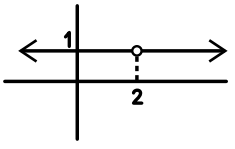
Answers:

1) 

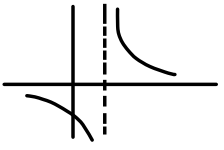
f is not defined at $x_o = 0$ (finite jump)

2) 

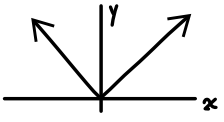
x = -1 is an isolated point of g. (isolated point)

3) 

h is undefined at $x_o = 2$. h having limit (=1) at $x_o = 2$ the discontinuity is removable.

4) 
k is undefined at $x_o = 1$ (infinite jump)

Example 2. Test the function $f(x) = |x|$ for continuity at the origin.



Solution. Since $\lim_{x \to o} |x| = 0$ and this limit is equal to $f(0)$, f is continuous at 0.

Example 3. Test the function $f(x) = [\![3x + 1]\!]$ at $x_o = \frac{1}{2}$.

Solution. $\lim_{x \to \frac{1}{2}} f(x) = 2 = f(\frac{1}{2})$.

It is continuous.

